Exercise 6B

1 a $\begin{vmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{vmatrix} = 1\begin{vmatrix}2 & 0\\0 & 3\end{vmatrix} - 0\begin{vmatrix}0 & 0\\0 & 3\end{vmatrix} + 0\begin{vmatrix}0 & 2\\0 & 0\end{vmatrix}$

$= 6 - 0$
$= 6$

b $\begin{vmatrix}0 & 4 & 0\\5 & -2 & 3\\2 & 1 & 4\end{vmatrix} = 0\begin{vmatrix}-2 & 3\\1 & 4\end{vmatrix} - 4\begin{vmatrix}5 & 3\\2 & 4\end{vmatrix} + 0\begin{vmatrix}5 & -2\\2 & 1\end{vmatrix}$

$= -4(20-6)$
$= -56$

c $\begin{vmatrix}1 & 0 & 1\\2 & 4 & 1\\3 & 5 & 2\end{vmatrix} = 1\begin{vmatrix}4 & 1\\5 & 2\end{vmatrix} - 0\begin{vmatrix}2 & 1\\3 & 2\end{vmatrix} + 1\begin{vmatrix}2 & 4\\3 & 5\end{vmatrix}$

$= (8-5) + (10-12)$
$= 1$

d $\begin{vmatrix}2 & -3 & 4\\2 & 2 & 2\\5 & 5 & 5\end{vmatrix} = 2\begin{vmatrix}2 & 2\\5 & 5\end{vmatrix} + 3\begin{vmatrix}2 & 2\\5 & 5\end{vmatrix} + 4\begin{vmatrix}2 & 2\\5 & 5\end{vmatrix}$

$= 2(10-10) + 3(10-10) + 4(10-10)$
$= 0$

2 a $\begin{vmatrix}4 & 3 & -1\\2 & -2 & 0\\0 & 4 & -2\end{vmatrix} = 4\begin{vmatrix}-2 & 0\\4 & -2\end{vmatrix} - 3\begin{vmatrix}2 & 0\\0 & -2\end{vmatrix} - 1\begin{vmatrix}2 & -2\\0 & 4\end{vmatrix}$

$= 4(4-0) - 3(-4-0) - 1(8+0)$
$= 16 + 12 - 8$
$= 20$

b $\begin{vmatrix}3 & -2 & 1\\4 & 1 & -3\\7 & 2 & -4\end{vmatrix} = 3\begin{vmatrix}1 & -3\\2 & -4\end{vmatrix} + 2\begin{vmatrix}4 & -3\\7 & -4\end{vmatrix} + 1\begin{vmatrix}4 & 1\\7 & 2\end{vmatrix}$

$= 3(-4+6) + 2(-16+21) + 1(8-7)$
$= 6 + 10 + 1$
$= 17$

2 c $\begin{vmatrix} 5 & -2 & -3 \\ 6 & 4 & 2 \\ -2 & -4 & -3 \end{vmatrix} = 5\begin{vmatrix} 4 & 2 \\ -4 & -3 \end{vmatrix} + 2\begin{vmatrix} 6 & 2 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 6 & 4 \\ -2 & -4 \end{vmatrix}$

$= 5(-12+8)+2(-18+4)-3(-24+8)$
$= -20-28+48$
$= 0$

3 $\begin{vmatrix} 2 & 1 & -4 \\ 2k+1 & 3 & k \\ 1 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & k \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 2k+1 & k \\ 1 & 1 \end{vmatrix} - 4\begin{vmatrix} 2k+1 & 3 \\ 1 & 0 \end{vmatrix}$

$= 2(3-0)-1(2k+1-k)-4(0-3)$
$= 6-k-1+12$
$= 17-k$

Since the determinant is singular:
$17-k=0$
$k=17$

4 $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{pmatrix}$

$\det(\mathbf{A}) = \begin{vmatrix} 2 & -1 & 3 \\ k & 2 & 4 \\ -2 & 1 & k+3 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & 4 \\ 1 & k+3 \end{vmatrix} + 1\begin{vmatrix} k & 4 \\ -2 & k+3 \end{vmatrix} + 3\begin{vmatrix} k & 2 \\ -2 & 1 \end{vmatrix}$
$= 2(2k+6-4)+1(k^2+3k+8)+3(k+4)$
$= 4k+4+k^2+3k+8+3k+12$
$= k^2+10k+24$

$\det(\mathbf{A}) = 8$
Therefore:
$k^2+10k+24=8$
$k^2+10k+16=0$
$(k+2)(k+8)=0$
$k=-2$ or $k=-8$

5 a $\mathbf{A}=\begin{pmatrix}2&5&3\\-2&0&4\\3&10&8\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1&1&0\\1&2&2\\0&-2&-1\end{pmatrix}$

$$\det(\mathbf{A})=\begin{vmatrix}2&5&3\\-2&0&4\\3&10&8\end{vmatrix}$$
$$=2\begin{vmatrix}0&4\\10&8\end{vmatrix}-5\begin{vmatrix}-2&4\\3&8\end{vmatrix}+3\begin{vmatrix}-2&0\\3&10\end{vmatrix}$$
$$=2(0-40)-5(-16-12)+3(-20-0)$$
$$=-80+140-60$$
$$=0$$

Therefore, **A** is singular.

b $\mathbf{AB}=\begin{pmatrix}2&5&3\\-2&0&4\\3&10&8\end{pmatrix}\begin{pmatrix}1&1&0\\1&2&2\\0&-2&-1\end{pmatrix}$

$$=\begin{pmatrix}2+5+0&2+10-6&0+10-3\\-2+0+0&-2+0-8&0+0-4\\3+10+0&3+20-16&0+20-8\end{pmatrix}$$
$$=\begin{pmatrix}7&6&7\\-2&-10&-4\\13&7&12\end{pmatrix}$$

c $\det(\mathbf{AB})=\begin{vmatrix}7&6&7\\-2&-10&-4\\13&7&12\end{vmatrix}$

$$=7\begin{vmatrix}-10&-4\\7&12\end{vmatrix}-6\begin{vmatrix}-2&-4\\13&12\end{vmatrix}+7\begin{vmatrix}-2&-10\\13&7\end{vmatrix}$$
$$=7(-120+28)-6(-24+52)+7(-14+130)$$
$$=-644-168+812$$
$$=0$$

Therefore, **AB** is singular.

6 a $\mathbf{A}=\begin{pmatrix}4 & 5 & -2\\ 2 & -3 & 2\\ 2 & -4 & 3\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{A})&=\begin{vmatrix}4 & 5 & -2\\ 2 & -3 & 2\\ 2 & -4 & 3\end{vmatrix}\\ &=4\begin{vmatrix}-3 & 2\\ -4 & 3\end{vmatrix}-5\begin{vmatrix}2 & 2\\ 2 & 3\end{vmatrix}-2\begin{vmatrix}2 & -3\\ 2 & -4\end{vmatrix}\\ &=4(-9+8)-5(6-4)-2(-8+6)\\ &=-4-10+4\\ &=-10\end{aligned}$$

b $\mathbf{A}=\begin{pmatrix}4 & 5 & -2\\ 2 & -3 & 2\\ 2 & -4 & 3\end{pmatrix}\Rightarrow\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}4 & 2 & 2\\ 5 & -3 & -4\\ -2 & 2 & 3\end{pmatrix}$

c $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}4 & 2 & 2\\ 5 & -3 & -4\\ -2 & 2 & 3\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{A}^{\mathrm{T}})&=\begin{vmatrix}4 & 2 & 2\\ 5 & -3 & -4\\ -2 & 2 & 3\end{vmatrix}\\ &=4\begin{vmatrix}-3 & -4\\ 2 & 3\end{vmatrix}-2\begin{vmatrix}5 & -4\\ -2 & 3\end{vmatrix}+2\begin{vmatrix}5 & -3\\ -2 & 2\end{vmatrix}\\ &=4(-9+8)-2(15-8)+2(10-6)\\ &=-4-14+8\\ &=-10\end{aligned}$$

Therefore, $\det(\mathbf{A}^{\mathrm{T}})=\det(\mathbf{A})$ as required

7 a $\mathbf{A}=\begin{pmatrix}0 & a & -b\\ -a & 0 & c\\ b & -c & 0\end{pmatrix}$

$$\begin{aligned}\begin{vmatrix}0 & a & -b\\ -a & 0 & c\\ b & -c & 0\end{vmatrix}&=0\begin{vmatrix}0 & c\\ -c & 0\end{vmatrix}-a\begin{vmatrix}-a & c\\ b & 0\end{vmatrix}-b\begin{vmatrix}-a & 0\\ b & -c\end{vmatrix}\\ &=0(0+c^2)-a(0-bc)-b(ac-0)\\ &=0+abc-abc\\ &=0\end{aligned}$$

Therefore, **A** is singular for all values of a, b and c.

7 b $\mathbf{B} = \begin{pmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{pmatrix}$

$$\begin{vmatrix} 2 & -2 & 4 \\ 3 & x & -2 \\ -1 & 3 & x \end{vmatrix} = 2\begin{vmatrix} x & -2 \\ 3 & x \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ -1 & x \end{vmatrix} + 4\begin{vmatrix} 3 & x \\ -1 & 3 \end{vmatrix}$$

$$= 2\left(x^2+6\right)+2\left(3x-2\right)+4\left(9+x\right)$$
$$= 2x^2+12+6x-4+36+4x$$
$$= 2x^2+10x+44$$
$$= 2\left(x^2+5x+22\right)$$
$$= 2\left[\left(x+\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+22\right]$$
$$= 2\left[\left(x+\frac{5}{2}\right)^2+\frac{63}{4}\right]$$

Therefore, for any real value of x:

$$\det \mathbf{B} \geq 2 \times \frac{63}{4} > 0$$

As det($\mathbf{B}$) $\neq$ 0, $\mathbf{B}$ cannot be singular for any real value of x.

8 $\mathbf{A} = \begin{pmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{pmatrix}$

$$\begin{vmatrix} x-3 & -2 & 0 \\ 1 & x & -2 \\ -2 & -1 & x+1 \end{vmatrix} = \left(x-3\right)\begin{vmatrix} x & -2 \\ -1 & x+1 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -2 & x+1 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ -2 & -1 \end{vmatrix}$$

$$= \left(x-3\right)\left(x^2+x-2\right)+2\left(x+1-4\right)+0\left(-1+2x\right)$$
$$= x^3+x^2-2x-3x^2-3x+6+2x-6$$
$$= x^3-2x^2-3x$$
$$= x\left(x^2-2x-3\right)$$
$$= x\left(x+1\right)\left(x-3\right)$$

If $\mathbf{A}$ is singular then det($\mathbf{A}$) = 0, therefore:

$$x\left(x+1\right)\left(x-3\right)=0$$

$x = 0$, $x = -1$ or $x = 3$